Fuzzy Logic

Created by
Omar Alkhateeb

For more MindMaps
Check our Blitz Website
https://blitz.deveopix.com/

Fuzzy Neural and Neural Fuzzy Systems

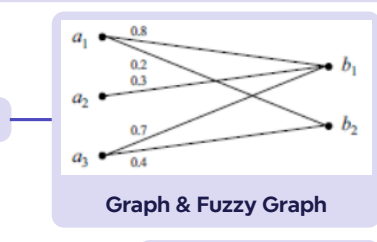
- in this lecture i'll try something new you can try too
  - i sent it to qwen.ai
  - and tell him to explain it
  - i'll create the mindmap of this lecture from his explaination
- **Overview of Fuzzy-Neural Systems**
  - **Objective** — enhance the knowledge representation capability of conventional neural networks — by incorporating fuzzy concepts
  - **Approach**
    - Introduce fuzzy concepts at various levels of a neural network
      - inputs
      - weights
      - aggregation operations
      - activation functions
      - outputs
    - Replace standard mathematical models for neurons with fuzzy neurons
      - using t-norms (for AND operations)
      - t-conorms (for OR operations)
      - to build aggregation operations
        - This results in a fuzzy-neural system
        - that can handle fuzzy inputs and uses an analog of the conventional backpropagation algorithm for training
- **Overview of Neuro-Fuzzy Systems**
  - **Objective** — Use neural networks to provide learning capabilities to fuzzy systems
  - **Approach**
    - Neural networks are used in fuzzy modeling — and control to automate the design and tuning of fuzzy systems
    - This leads to a neuro-fuzzy system — which is a fuzzy system represented as a modified neural network
    - The resulting system combines the interpretability of fuzzy logic — with the learning capabilities of neural networks
- **Key Differences Between Fuzzy Systems and Neural Networks**
  - **Fuzzy Systems**
    - More user-friendly — because their behavior can be explained using fuzzy rules based on human reasoning
    - Can encode expert knowledge directly — using linguistic labels but require significant effort to design and tune membership functions
    - Limited to applications where expert knowledge is available — and the number of input variables is small
  - **Neural Networks**
    - Good at handling raw data and learning from it but lack transparency
    - Can automate the design and tuning of fuzzy systems
      - reducing development time
      - and improving performance
- **Fuzzy Concepts in Neural Networks** — three types of fuzzy neurons
  - Type I (min-max fuzzy neuron)
  - Type II (product-sum fuzzy neuron)
  - Type III (max-min fuzzy neuron)
  - These fuzzy neurons allow neural networks to handle fuzzy inputs and outputs — enhancing their ability to represent complex relationships
- **Learning and Adaptation for Fuzzy Neurons**
  - **Learning Process**
    - Similar to traditional neural networks — fuzzy neurons can be trained using backpropagation-like algorithms
    - The weights and parameters of fuzzy neurons are adjusted — to minimize the error between the predicted output and the target output
  - **Saturating Function** — A unipolar saturating function is often used — to ensure that the output of fuzzy neurons remains within a specific range (e.g., [0, 1])
- **Regular Fuzzy Neural Networks (RFNNs)**
  - RFNNs combine the structure of neural networks with fuzzy logic principles
  - They typically have multiple layers
    - **Input layer** — Receives crisp inputs
    - **Fuzzification layer** — Converts crisp inputs into fuzzy sets
    - **Rule layer** — Represents fuzzy rules
    - **Output membership layer** — Combines rule outputs using union operations
    - **Defuzzification layer** — Converts the final fuzzy output into a crisp value
- **Neuro-Fuzzy Architecture** — The neuro-fuzzy architecture is similar to a multi-layer neural network but includes layers specifically designed for fuzzy inference
  - **Layer 1 (Input Layer)** — Transmits external crisp signals directly to the next layer
  - **Layer 2 (Fuzzification Layer)**
    - Represents fuzzy sets used in the antecedents of fuzzy rules
    - Each neuron corresponds to a fuzzy set and determines the degree to which an input belongs to that set.
    - Triangular membership functions are commonly used, defined by parameters a and b — $\mu(x)=\begin{cases}\frac{x-a}{b-a}, & \text{if } a\le x\le b\\ \frac{b-x}{b-a}, & \text{if } b\le x\le c\\ 0, & \text{otherwise}\end{cases}$
  - **Layer 3 (Fuzzy Rule Layer)**
    - Each neuron represents a single fuzzy rule
    - Implements the intersection of fuzzy sets in the antecedents using the product operator
  - **Layer 4 (Output Membership Layer)**
    - Represents fuzzy sets in the consequents of fuzzy rules
    - Implements the union of rule outputs using the probabilistic OR
  - **Layer 5 (Defuzzification Layer)**
    - Combines the output fuzzy sets into a single crisp value
    - Commonly uses the centroid method or sum-product composition
- **Adaptive Neuro-Fuzzy Inference Systems (ANFIS)**
  - **Purpose** — Automate the design and tuning of fuzzy systems — using a hybrid learning algorithm
  - **Architecture**
    - Based on Sugeno-type fuzzy models — (typically first-order or zero-order)
    - Consists of multiple layers — similar to the neuro-fuzzy architecture described above
    - Uses a combination of least-squares estimation — and backpropagation for parameter tuning
  - **Advantages**
    - Combines the interpretability of fuzzy systems — with the learning capabilities of neural networks
    - Suitable for modeling nonlinear systems — and performing tasks like
      - classification
      - regression

Fuzzy Pattern Recognition

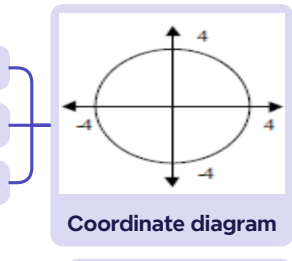
- **pattern recognition**
  - its **purpose**
    - to assign each input to one of c possible pattern classes (or data clusters)
    - different input observations — should be assigned to the same class — if they have similar features
  - **pattern**
    - an abstraction
    - represented by a set of measurements — describing a **physical** object
  - **pattern class (category)** — set of patterns sharing common attributes
  - & other stuff about supervised learning & unsupervised learning mentioned in the ML diagram, check our website for it (it has just some fuzzy laws that we already discussed, but i don't have time to rewrite everything)

Fuzzy Relations

- **Classical** vs **Fuzzy** Relations
  - in crisp relations — the elements may
    - completely relate
    - or not
  - in fuzzy relations — we have **infinite** number of relationships — between the completely related and not related
- **Cartesian Product of Relation**
  - Cartesian product of sets A1,A2,...,Ar — is the set of all ordered r-tuples (a1,a2,...,ar) — where ai ∈ Ai
  - denoted as — A1×A2×⋯×Ar
  - If all sets are identical — Cartesian product is written as A^r
- **Crisp relations**
  - A crisp relation among classical sets X and Y — is a subset of their Cartesian product X×Y — Binary relations are most commonly used
  - Strength of the relationship — between ordered pairs of elements — is measured using the characteristic function denoted by χ
  - example — if we have $X=\{2,4,6\}$ and $Y=\{p,q,r\}$ — if both sets are related to each other entirely
    - then their relation matrix $R=\begin{array}{c|ccc} & p & q & r\\ \hline 2 & 1 & 1 & 1\\ 4 & 1 & 1 & 1\\ 6 & 1 & 1 & 1\end{array}$
    - general crisp relation $\chi_R(x,y)=\begin{cases}1, & (x,y)\in R\\ 0, & (x,y)\notin R\end{cases}$
  - operation on crisp relations
    - **Union** — Max of both
    - **Intersection** — Min of both
    - **Complement** — 1 - relation
    - **Containment** — one is a subset of the other
    - **Identity** — leaves the input unchanged
- **Fuzzy Relations**
  - A fuzzy relation R is a 2D MF $R=\{((x,y),\mu_R(x,y))\,|\,(x,y)\in X\times Y\}$
  - **Max-Min Composition** — method used to combine two fuzzy relations
    - R (defined on X×Y)
    - and S (defined on Y×Z)
    - into a new fuzzy relation R∘S defined on X×Z $\mu_{R\circ S}(x,z)=\max_y[\min(\mu_R(x,y),\mu_S(y,z))]$
  - **Max-Product Composition** — method used to combine two fuzzy relations
    - R (defined on X×Y)
    - and S (defined on Y×Z)
    - into a new fuzzy relation R∘S defined on X×Z $\mu_{R\circ S}(x,z)=\max_v(\mu_R(x,v)\cdot\mu_S(v,z))$
  - **Defintion**
    - mapping from the Cartesian space X x Y to the interval [0,1]
    - **strength** of the mapping — is expressed by the membership function of the relation μR(x,y)
  - **Operations** on fuzzy relations
    - **Union** — Max of both
    - **Intersection** — Min of both
    - **Complement** — 1 - membership function values
    - **Containment** — membership function values of one is included in the membership function values of the other
  - **properties** of fuzzy relations
    - **Commutativity** — order of operands does not affect the result
    - **Associativity** — grouping of operands does not affect the result
    - **Distributivity** — Operations distribute over each other
    - **Idempotency** — Applying an operation to a set with itself does not change the set
    - **Involution** — Applying an operation twice returns the original set
    - **Axiom of the Excluded Middle** — fuzzy relation and its complement do not cover all possibilities completely
    - **Axiom of the Contradiction** — fuzzy relation and its complement do not overlap completely
  - **Fuzzy Cartesian Product** — product of 2 fu Floating Topic — which is membership function that takes min(of both membership functions)
  - **Max-Average Composition** $S\underset{\sim}{}\ \underset{\sim\,\text{org}\,\sim}{O}\ R(x,z)=\left\{(x,z),\frac{1}{2}\max\left(\mu_{\underset{\sim}{R}}(x,y)+\mu_{\underset{\sim}{S}}(y,z)\right)\Big/ x\in X,y\in Y,z\in Z\right\}$
- **Representation Methods of Relations**

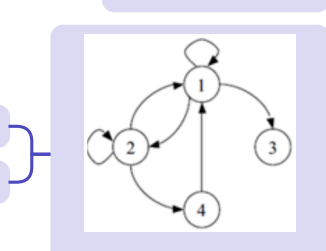


  - **Graph & Fuzzy Graph** — which is G = (V, E)
    - V — set of vertices
    - E
      - in normal graph — set of edges
      - in fuzzy graph — fuzzy set of edges between vertices

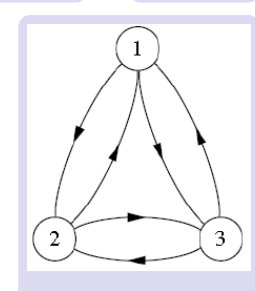


  - **Coordinate diagram**
    - plotting members of **A** on **x axis**
    - and that of **B** on **y axis**
    - and then the members of **A × B** lie on the **space**
  - **Matrix** — by manipulating relation matrix $m_{ij}=\begin{cases}1, & (a_i,b_j)\in R & i=1,2,3,\ldots,m\\ 0, & (a_i,b_j)\notin R & j=1,2,3,\ldots,n\end{cases}$

R	b_1	b_2	b_3
a_1	1	0	0
a_2	0	1	0
a_3	0	1	0
a_4	0	0	1



  - **Digraph (directed graph)**
    - Elements are represented as nodes
    - relations between elements as directed edges
- **Equivalence relation** must have
  - **Reflexivity** — every vertex in the graph originates a single loop



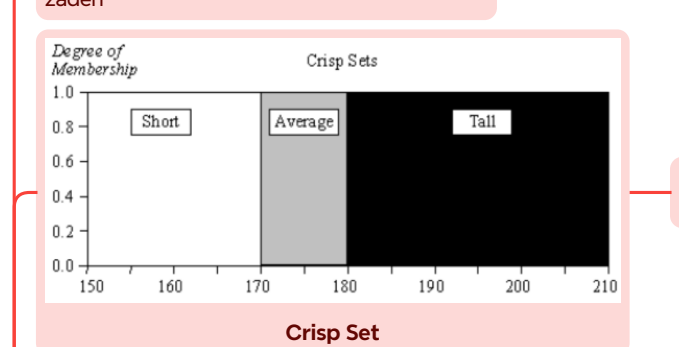
  - **Symmetry**
  - **Transitivity** $(x,y)\in R,(y,z)\in R\rightarrow(x,z)\in R$
- **Special Relations**
  - **Compaitability** — if we applied compaitibility relation to a set — then we can decompose the set into disjoint subsets
  - **Order Relation** — must be binary relation that is
    - Reflexive
    - Antisymmetric
    - Transitive
  - **Pre-order Relation** — must be
    - Reflexive
    - Transitive
- Fuzzy Matrix **Operations**
  - sum — take max
  - max-min — take min then max
  - max-product — multiply then take max
  - scalar product — multiply with all

Fuzzy Number

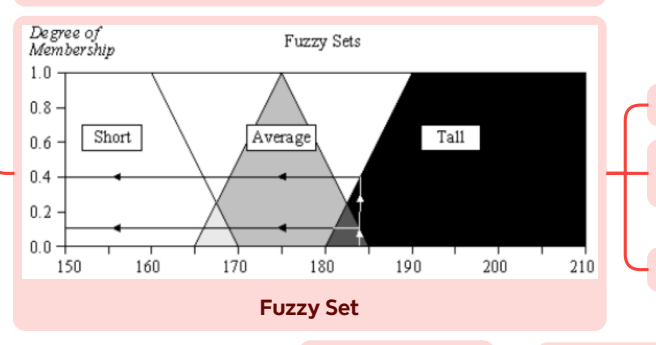
- **Concepts** of fuzzy numbers
  - **Interval number**
    - can be expressed as interval — A = [a1, a2, a3]
    - can be expressed as membership function $\mu_A(x)=\begin{cases}0, & x<a_1\\ 1, & a_1\le x\le a_3\\ 0, & x>a_3\end{cases}$
  - when to call it fuzzy number?
    - **fuzzy set** of R — R = **Re**al number
    - **convex**
    - **normal** — we have a value where there is 100% membership of a value
    - the membership function **values** must be **[0, 1]**
    - membership function must be **continuous**
- Positivity & Negativity
  - positive if — zero membership values when — x <= 0
  - negative if — zero membership values when — x >= 0
- **L-R** Fuzzy Number — described by 2 functions
  - L — for left spread
  - R — for right spread
- **α-cut** of a fuzzy number — interval that contains all the values x — where the membership value μA(x) is **greater than** or equal to a specified level α
- **Operations on intervals**
  - **addition**
  - **subtraction**
    - each element with the same index element of the other interval
  - **multplication**
    - we make AND operation with each element multiplied by all other interval elements
    - ,
    - we make OR operation with each element multiplied by all other interval elements
  - **Division** — same as multiplication but division
  - **Inverse interval** — we inverse each element but with AND and OR operations with other inverted elements
- operations on **α-cut** intervals — same as intervals
- operations on fuzzy numbers
  - **addition** $\mu_{A(+)B}(z)=\bigvee_{z=x+y}(\mu_A(x)\wedge\mu_B(y))$
    - it's determined by considering all pairs so z=x+y
    - For each pair (x,y), compute the minimum (logical AND) of the membership values μA(x) and μB(y)
    - Take the maximum (logical OR) of all these minimum values across all pairs (x,y)
    - This gives the overall membership value μ(A(+)B)(z)
  - **subtraction** $\mu_{A(-)B}(z)=\bigvee_{z=x-y}(\mu_A(x)\wedge\mu_B(y))$
    - it's determined by considering all pairs so z=x-y
    - For each pair (x,y), compute the minimum (logical AND) of the membership values μA(x) and μB(y)
    - Take the maximum (logical OR) of all these minimum values across all pairs (x,y)
  - **multiplication** $\mu_{A(\bullet)B}(z)=\bigvee_{z=x\bullet y}(\mu_A(x)\wedge\mu_B(y))$ — like above but considering all pairs so z=x.y
  - **Division** $\mu_{A(/)B}(z)=\bigvee_{z=x/y}(\mu_A(x)\wedge\mu_B(y))$ — like above but considering all pairs so z=x/y
  - **Mininmum** $\mu_{A(\wedge)B}(z)=\bigvee_{z=x\wedge y}(\mu_A(x)\wedge\mu_B(y))$
  - **Maximum** $\mu_{A(\vee)B}(z)=\bigvee_{z=x\vee y}(\mu_A(x)\wedge\mu_B(y))$
  - Multiplication of a **scalar** value to the interval: $a[b_1,b_3]=[a\bullet b_1\wedge a\bullet b_3,\ a\bullet b_1\vee a\bullet b_3]$
  - Multiplication of a **scalar** value to **α-cut** interval: $a[b_1^{(\alpha)},b_3^{(\alpha)}]=[a\bullet b_1^{(\alpha)}\wedge a\bullet b_3^{(\alpha)},\ a\bullet b_1^{(\alpha)}\vee a\bullet b_3^{(\alpha)}]$
- **Tringular Fuzzy Number**
  - A = (a1 , a 2, a 3)
  - its membership function $\mu_{(A)}(x)=\begin{cases}0, & x<a_1\\ \frac{x-a_1}{a_2-a_1}, & a_1\le x\le a_2\\ \frac{a_3-x}{a_3-a_2}, & a_2\le x\le a_3\\ 0, & x>a_3\end{cases}$
  - **α-cut** interval of triangular fuzzy number $\frac{a_1^{(\alpha)}-a_1}{a_2-a_1}=\alpha \quad \frac{a_3-a_3^{(\alpha)}}{a_3-a_2}=\alpha$
    - thus $A_\alpha=[a_1^{(\alpha)},a_3^{(\alpha)}]=[(a_2-a_1)\alpha+a_1,\ -(a_3-a_2)\alpha+a_3]$
  - Operations on tringular fuzzy numbers
    - **addition & subtraction** — results in triangular fuzzy number
    - **multiplication & division** — doesn't result in triangular fuzzy number
    - **max & min** — doesn't result in triangular fuzzy number
- **Trapezoidal Fuzzy Number**
  - A = (a1, a2, a3, a4)
  - its membership function $\mu_A(x)=\begin{cases}0, & x<a_1\\ \frac{x-a_1}{a_2-a_1}, & a_1\le x\le a_2\\ 1, & a_2\le x\le a_3\\ \frac{a_4-x}{a_4-a_3}, & a_3\le x\le a_4\\ 0, & x>a_4\end{cases}$
  - Operations on tringular fuzzy numbers
    - **addition & subtraction** — results in triangular fuzzy number
    - **multiplication & division** — doesn't result in triangular fuzzy number
    - **max & min** — doesn't result in triangular fuzzy number

Introduction

- the fuzzy logic science was proposed by lotfi zadeh



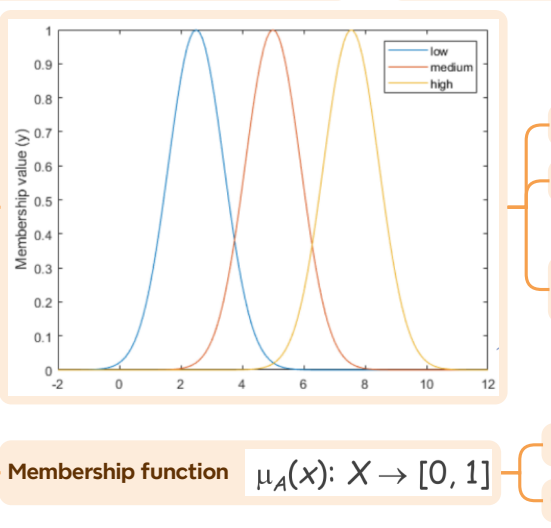
- **Crisp Set** — container that wholly **includes** or wholly **excludes** any given element



- **Fuzzy Set**
  - a set without crisp, clearly defined boundaries
  - container that wholly includes or wholly excludes any given element
  - we have **uncertainty**
    - because of non-sharp boundaries
    - not because of lack of information
- **Fuzzy Control System**
  1. **Fuzzification** — making a crisp quantity fuzzy
  2. **Inferencing** — Evaluating all rules and determining their truth values
  3. **Defuzzification** — Converting a fuzzy set into a crisp output

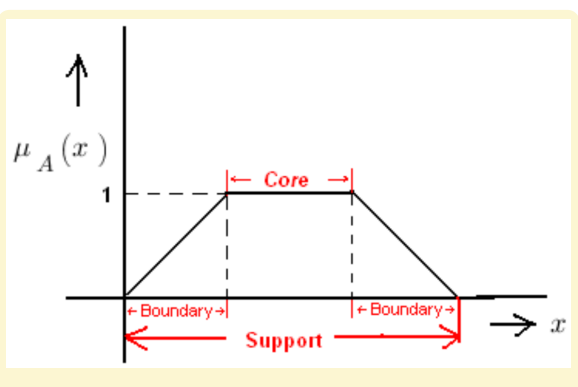
Sets

- **Definition**
  - an unordered collection of objects
  - objects in set are called elements or members of set S — S contains its elements
- **Defining Sets**
  - **List Method**
    - by directly listing all its elements
    - ex: S = {2, 3, 5, 7, 11, 13, 17, 19}
  - **Rule Method**
    - by describing the properties that its elements should satisfy
    - we use the notation {x E A | P(x)}
  - **Characteristic function**
    - set is defined by a function
    - ex: $\chi_A(x)=\begin{cases}1 & \text{for } x\in A\\ 0 & \text{for } x\notin A.\end{cases}$
- **Membership**
  - "The most basic question in set theory is whether an element is in a set"
  - the order and number of occurrence are not important
- **Cardinality** — the size of set
- **Family of Sets**
  - set whose elements are themselves sets
  - ex: A E {A1, A2, ..., An}
- **Subsets**
  - $A\subseteq B$ A is a **subset** of B — and we can say: A is **included** in B
  - $A=B$ A and B are **equal** sets
  - $A\neq B$ A and B are **not equal** sets
  - $A\subset B$ A is a **proper subset** of B
- **Basic Operations** on Sets
  - **intersection** $A\cap B=\{x\in U\mid x\in A \text{ and } x\in B\}$ — two sets are disjoint if their intersection is empty
  - **union** $A\cup B=\{x\in U\mid x\in A \text{ or } x\in B\}$ — we can represent union also $|A\cup B|=|A|+|B|-|A\cap B|$
  - **difference** $A-B=\{x\in U\mid x\in A \text{ and } x\notin B\}$ — we can represent difference also $|A-B|=|A|-|A\cap B|$
  - **Complement** $\overline{A}=A^c=\{x\in U\mid x\notin A\}$ — also if $A\subseteq B$, then $\overline{B}\subseteq\overline{A}$
- **Partitions of Sets** — A collection of nonempty sets {A1, A2, ..., An} is a partition of a set A — if and only if
  - $A=A_1\cup A_2\cup\cdots\cup A_n$
  - $A_1, A_2, \ldots, A_n$ are disjoint
- **Power Sets** — contains all the subsets of A as members
  - ex: pow({a,b}) = {Φ, {a}, {b}, {a,b}}
  - if A has **n** elements — then pow(A) has **2^n** elements
- **Cartesian Products** — all ordered pairs (a,b), where a is in A and b is in B $A\times B=\{(a,b)\mid a\in A \text{ and } b\in B\}$
- **Generalized Operations** on Sets
  - **Generalized Union** $\bigcup_{i\in I}A_i=\{x\mid x\in A_i \text{ for some } i\in I\}$
  - **Generalized Intersection** $\bigcap_{i\in I}A_i=\{x\mid x\in A_i \text{ for all } i\in I\}$
- **Set Identities**
  - $A\cap B\subseteq A$
  - $A\subseteq A\cup B$
  - if $A\subseteq B$ and $B\subseteq C$ — then $A\subseteq C$
  - $A\cap\overline{A}=\emptyset$
  - $\overline{\overline{A}}=A$
  - $A\cup B=(A-B)\cup(B-A)\cup(A\cap B)$
  - commutative property $A\cup B=B\cup A$, $A\cap B=B\cap A$
  - Associative property $A\cup(B\cup C)=(A\cup B)\cup C$, $A\cap(B\cap C)=(A\cap B)\cap C$
  - Distributive Law $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$ — or $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$
  - De Morgan's Law $\overline{A\cup B}=\overline{A}\cap\overline{B}$ — or $\overline{A\cap B}=\overline{A}\cup\overline{B}$
- **Representing Fuzzy Sets**

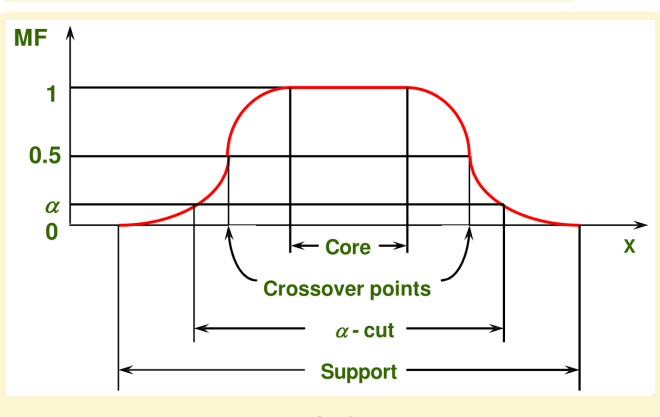


  - x-axis — is the range of all possible values
  - y-axis — membership value of the fuzzy set
  - or we can represent it like this $A=\{(x_1,0)(x_2,1)(x_3,1)(x_4,0)(x_5,0)\}$
    - where the first number in each pair represent the possible value
    - and the second number represents membership function of the element
  - **Membership function** $\mu_A(x): X\rightarrow[0,1]$
    - 1 if it's totally in the set
    - 0 if not in the set
    - if between them so it's partly in A
- **Linguistic Variables**
  - is a fuzzy variable
  - where we conclude rules from statements — ex: john is tall
    - variable john
    - value tall
  - **Hedges**
    - words that can modify the shape of fuzzy sets
    - **adverbs** like
      - very
      - somewhat
      - quite
      - more
      - less
      - slightly
- **Opertaions on fuzzy sets**
  - **Union** — take the maximum value between the variables
  - **Intersection** — take the minimum value between the variables
  - **Complement** — 1 - (value of membership to the set)
  - **Containment** — if one set is contained in another set — the degree of membership is lesser that or equal to the other set degree of membership

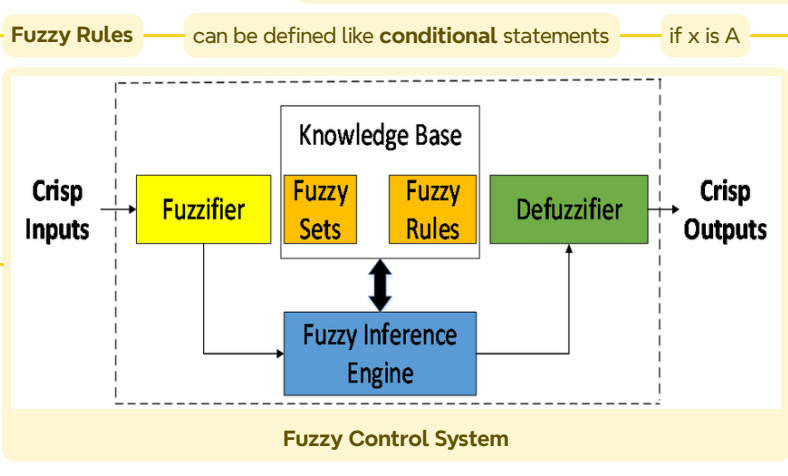
Basic concepts & Fuzzy rules



- **Features of Membership Function**
  - **Core** — elements that are totally in the set
  - **Support** — elements with non-zero membership
  - **Boundary** — elements with membership between]0, 1[



- **Terminology**
  - **Singleton** — fuzzy set whose support is single point
  - **Crossover Point** — point where its membership is 0.5
  - **Height** — largest membership value reached by any point
  - **alpha-cut** — all elements that have value greater than "alpha"
    - **Strong alpha-cut** — all elements that have value **strictly** greater than "alpha"
- **Convexity** — a fuzzy set is convex if — its membership function satisfies $\mu_A(\lambda x_1+(1-\lambda)x_2)\ge\min[\mu_A(x_1),\mu_A(x_2)]$
- **Properties of Fuzzy Sets**
  - **Equality** — if and only if they have the same membership function
  - **Inclusion** — a fuzzy set is included in another fuzzy set if its membership function is included in the other
  - **Cardinality** — SUM of the values of the membership functions $\sum\mu_A(x_i)$
  - **Math Operations**
    - **number x set** — we multiply the number by the membership values for each variable
    - **set ^ number** — we raise the number to the values
- **Membership Functions**
  - **Triangular membership function** $Triangular(x,a,b,c)=\max(\min(\frac{x-a}{b-a},\frac{c-x}{c-b}),0)$
  - **Trapezoidal membership function** $Trapezoidal(x,a,b,c,d)=\max(\min(\frac{x-a}{b-a},\frac{d-x}{d-c}),0)$
  - **Gaussian membership function** $Gaussian(x,c,\sigma)=e^{\frac{-1}{2}\left(\frac{x-c}{\sigma}\right)^2}$
- **Fuzzy Rules** — can be defined like **conditional** statements — if x is A — then y is B

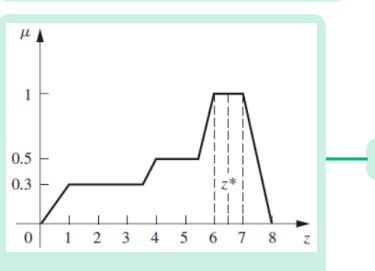


Fuzzy Control System

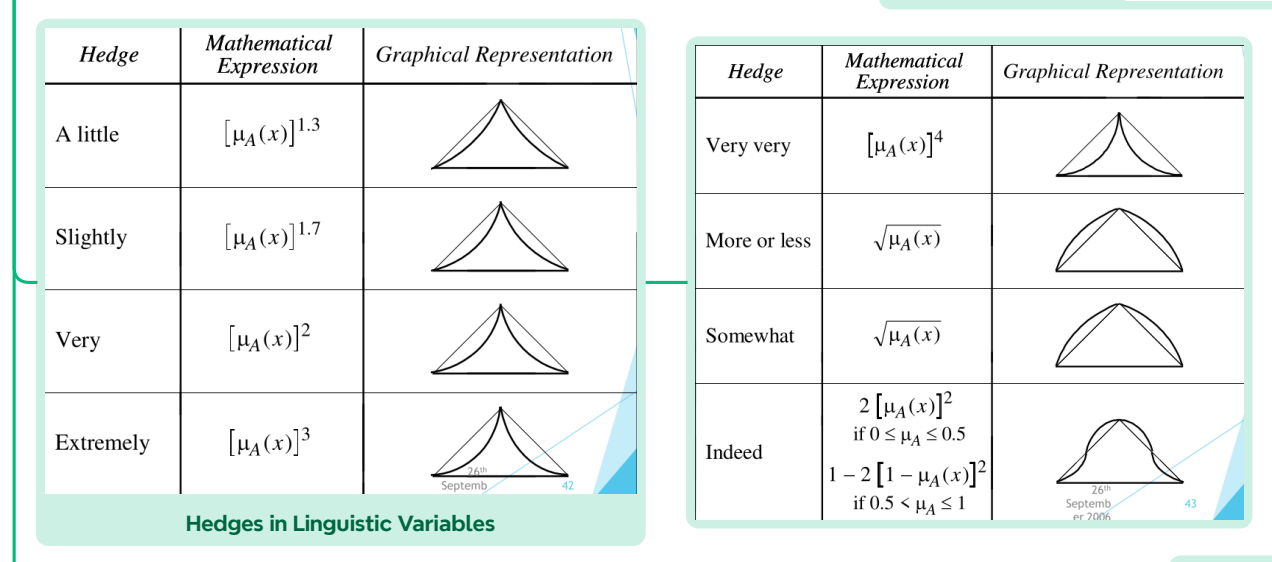
- **Architectures of Fuzzy Inference Systems** — **Mamdani Fuzzy Inference (which we'll use in this part but later we'll user other techniques too)**
  1. Fuzzification
     - to take the crisp inputs
     - then determine the degree to which these inputs belong to each of the appropriate fuzzy sets
  2. Rule Evaluation
     - take the fuzzified inputs
     - apply them to the antecedents of the fuzzy rules
     - **Clipping** — cut the consequent membership function at the level of the antecedent truth
     - **Scaling** — multiplying all its membership degrees by the truth value of the rule antecedent
  3. Aggregation — process of unification of the outputs of all rules — so it's combining everything in a single fuzzy set for each output variable
  4. Defuzzification
     - the input is the aggregated fuzzy set
     - where the output is a single number

Fuzzy Inference Techniques

- **Review** — any Fuzzy Inference System has these stages
  - **Fuzzification**
  - **Inferencing**
    - Rule base
    - Database (or dictionary)
    - reasoning mechanism
  - **Defuzzification** — extraction of a crisp value that best represents a fuzzy set
- **Mamdani fuzzy inference**
  - If <antecedence> then <consequence> — both **antecedence** & **consequence** are **Fuzzy**
  - its **defuzzifier**
    - Converts the fuzzy output of the inference engine to crisp
    - using membership functions similar to the ones used by the fuzzifier
  - **Five** commonly used **defuzzifying methods**
    - **Centroid of area (COA)** $COG=\frac{\sum_a^b\mu_A(x)x}{\sum_a^b\mu_A(x)}$ — also called weighted average method
    - **Bisector of area (BOA)** $\int_\alpha^{z_{BOA}}\mu_A(z)dz=\int_{z_{BOA}}^{\beta}\mu_A(z)dz,$



    - **Mean of maximum (MOM)** — we take the mean of maximum common values
    - **Smallest of maximum (SOM)**
    - **Largest of maximum (LOM)**
- **Sugeno fuzzy inference**
  - also known as TSK model
  - its **goal** — Generation of fuzzy rules from a given input-output data set
  - Sugeno style fyzzy rule
    - if x is A
    - & y is B
    - then z is f(x, y)
      - where z = f(x, y) is crisp function — often **polynomial**
      - while A & B are fuzzy sets
  - **zero-order sugeno model**
    - the most common
    - applies rules as follows
      - if x is A
      - & y is B
      - then z is k — where k is constant
  - **weighted average** — sum of each value multiplied by the variable / sum of values
  - comparison between **mamdani & sugeno**
    - **mamdani** — accepted for capturing expert knowledge
    - **sugeno**
      - works well with optimization and adaptive techniques
      - so good in control problems — or dynamic non-linear systems
- **Tsukamoto Fuzzy inference** — the consequent of each rule is represented by — a **monotonical MF** like $z=\frac{w_1z_1+w_2z_2}{w_1+w_2}$

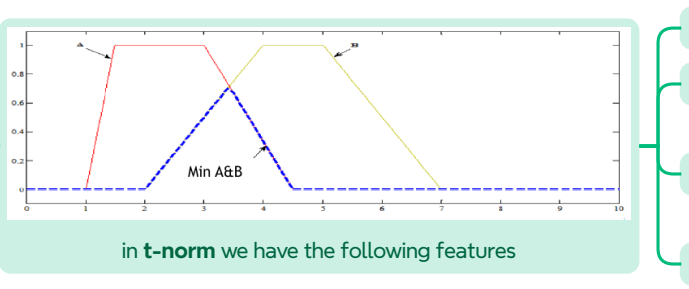


Hedge	Mathematical Expression	Graphical Representation
A little	$[\mu_A(x)]^{1.3}$	
Slightly	$[\mu_A(x)]^{1.7}$	
Very	$[\mu_A(x)]^2$	
Extremely	$[\mu_A(x)]^3$	

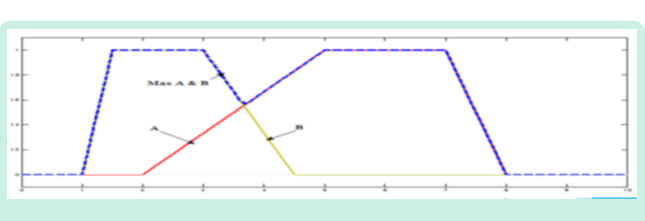
Hedge	Mathematical Expression	Graphical Representation
Very very	$[\mu_A(x)]^4$	
More or less	$\sqrt{\mu_A(x)}$	
Somewhat	$\sqrt{\mu_A(x)}$	
Indeed	$2[\mu_A(x)]^2$ if $0\le\mu_A\le0.5$ $1-2[1-\mu_A(x)]^2$ if $0.5<\mu_A\le1$	

Hedges in Linguistic Variables

- **Generalized Intersection (t-norm)**
Generalized Union (t-conorm/s-norm)



  - in **t-norm** we have the following features
    - **Symmetry** — T(x, y) = T(y, x)
    - **Associativity** — T(T(x , y), z) = T (x, T(y, z))
    - **monotonicity** — if
      - x1 <= x2
      - y1 <= y2
      - T(x1, y1) <= T(x2, y2)
    - **border condition** — T(x, 1) = x



  - in **s-norm** we have the following features
    - **Symmetry** — S(x, y) = S(y, x)
    - **Associativity** — S(S(x , y), z) = S(x, S(y, z))
    - **monotonicity** — if
      - x1 <= x2
      - y1 <= y2
      - S(x1, y1) <= S(x2, y2)
    - **border condition** — S(x, 0) = x
  - also we have **other features**
    - **min/max**
      - T(a , b) = min(a, b) = a ∧ b
      - S(a, b) = max(a , b) = a ∨ b
    - **Lukasiewicz**
      - T(a , b) = max(a+b-1, 0) = LAND (a , b)
      - S(a , b) = min(a+b, 1) = LOR (a, b)
    - **Probabilistic**
      - T(a , b) = ab = PAND(a , b)
      - S(a , b) = a + b - ab = POR(a, b)

Presented with xmind